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Supergravity flows and D-brane stability

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Abstract: We investigate the correspondence between existence/stability of BPS states in type-II string theory compactified on a Calabi-Yau manifold and BPS solutions of four-dimensional $N = 2$ supergravity. Some paradoxes emerge, and we propose a resolution by considering composite configurations. This in turn gives a smooth effective field theory description of decay at marginal stability. We also discuss the connection with 3-pronged strings, the Joyce transition of special lagrangian submanifolds, and Π-stability.

Keywords: D-branes, Black Holes in String Theory.

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1. Introduction

An old and common idea in physics is that a particle makes its presence manifest via excitation of fields. If one puts a lot of particles together, one gets a macroscopic object, well described by classical physics, and correspondingly one expects the field excitations to be well described by a classical field theory. In particular, it seems obvious that wherever we trust this field theory as a good description of the low energy physics, a well-behaved solution to the field equations corresponding to that object should exist.

Type-II string theory compactified on a Calabi-Yau manifold is described at low energies by a four-dimensional $\mathcal{N} = 2$ supergravity theory coupled to massles vector- and hypermultiplets. Quantum corrections to these theories are relatively well under control, and yet they are remarkably rich in content, with various intriguing connections to non-trivial physics and mathematics.

When string perturbation theory can be trusted, massive charged BPS particles in these theories can be described by D-branes wrapping non-trivial supersymmetric cycles in the Calabi-Yau manifold, or more generally by boundary states of the conformal field theory describing the relevant string perturbaton theory. When the low energy supergravity theory can be trusted, the same objects can be described by solutions to the field equations of motion. It turns out that not all charges support BPS states in the string theory, and that not all charges have BPS solutions in the supergravity theory. Thus, in suitable regimes, one naturally expects some sort of correspondence between supergravity solutions and D-branes.

Such a correspondence, while physically quite plausible, is in its consequences highly non trivial. For instance, it would give rise to a number of powerful predictions about the existence of special lagrangian submanifolds in Calabi-Yau manifolds, and the existence of boundary states in conformal field theories. However, as we will show in this paper, closer examination of this supposed correspondence reveals some intriguing puzzles. In particular, it turns out that the traditional assumption of the particle as a source localized in a single point of space leads to inconsistencies. Fortunately, once again, string theory finds its way out, and an interesting resolution to this paradox emerges.

The outline of this paper is as follows. In section 2, we briefly review the relevant geometry underlying low energy type-IIB string theory compactified on a Calabi-Yau manifold. In section 3, the derivation of the attractor flow equations is revisited. We start from a duality invariant bosonic action, discuss an interpretation as a static string action, derive the spherically symmetric attractor flow equations in two different forms, and comment on a subtlety arising for vanishing cycles. In section 4, we analyze some properties of solutions, with special emphasis on conifold charges, leading to "empty holes", and a short discussion of equal charge multicenter solutions. Then we tackle the existence issue: the attractor flow turns out to break down when the central charge has a regular zero, and this leads to a natural conjecture on the existence of BPS states [3]. However, this natural conjecture leads to some puzzling paradoxes. This is illustrated in section 5, using the example of a certain BPS state at the Gepner point of the quintic, known to exist, but nevertheless having a regular zero of the central charge. A second puzzle is illustrated in the SU (2) Seiberg-Witten model. In section 6, we propose a resolution to these puzzles; the key is to consider composite configurations. A thought experiment brings us rather naturally to the required configurations, in a spherical shell approximation. A stability check is made using test particle probes, and a representation as composite flows is given, making direct contact with 3-pronged strings in a suitble rigid limit. A smooth effective field theory picture for decay at marginal stability emerges, and Joyce's stability conjecture for special lagrangian submanifolds is recovered. We briefly comment on Π-stability [40]. Finally, in section 7, the analysis of general stationary multicenter solutions is initiated. Some properties of solutions can be inferred directly from the equations of motion. In particular, the intrinsic angular momentum of the multicenter composites is computed. Section 8 summarizes our conlusions, and indicates some open problems.

2. Geometry of IIB/CY compactifications

To establish notation and our setup, let us briefly review the low energy geometry of type-IIB string theory compactified on a Calabi-Yau 3-fold. Some of the more technical elements of this section are only needed for the derivation of some more technical results further on.

We will follow the manifestly duality invariant formalism of [3]. Consider type-IIB string theory compactified on a Calabi-Yau manifold X. The four-dimensional low energy theory is $\mathcal{N} = 2$ supergravity coupled to $n_v = h^{1,2}$ massless abelian vectormultiplets and $n_h = h^{1,1} + 1$ massless hypermultiplets, where the $h^{i,j}$ are the Hodge numbers of X. The hypermultiplet fields will play no role in the following and are set to zero.

The vectormultiplet scalars are given by the complex structure moduli of X, and the lattice of electric and magnetic charges is identified with $H^3(X, \mathbb{Z})$, the lattice of integral harmonic 3-forms on X. The "total" electromagnetic field strength $\mathcal{F}$ is (up to normalisation convention) equal to the type-IIB self-dual five-form field strength, and is assumed to have values in $\Omega^2(M_4) \otimes H^3(X, \mathbb{Z})$, where $\Omega^2(M_4)$ denotes the space of 2-forms on the four-dimensional spacetime M_4. The usual components of the field strength are retrieved by picking a symplectic basis α^I, β_I of $H^3(X, \mathbb{Z})$:

$$\mathcal{F} = F^I \otimes \beta_I - G_I \otimes \alpha^I \,. \tag{2.1}$$

The total field strength satisfies the self-duality constraint:

$$\mathcal{F} = *_{10}\mathcal{F} \,, \tag{2.2}$$

where $*_{10}$ is the Hodge star operator on the ten-dimensional space time, which factorises on the $M_4 \times X$ compactification as $*_{10} = *_4 \otimes *_X$. To prevent overly heavy notation, we will also denote the Hodge dual in X by a hat, so for any form Γ on X:

$$\widehat{\Gamma} \equiv *_X \Gamma\,. \tag{2.3}$$

Note that this operation is moduli-dependent. The constraint (2.2) relates the F and G components in (2.1). The (source free) equation of motion and the Bianchi identity of the electromagnetic field are combined in the equation $d\mathcal{F} = 0$, implying locally the existence of a potential: $\mathcal{F} = d\mathcal{A}$.

The geometry of the vector multiplet moduli space, parametrized with n_v coordinates z^a, is special Kähler [4]. The (positive definite) metric

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K} \tag{2.4}$$

is derived from the Kähler potential

$$\mathcal{K} = -\ln\left(i \int_X \Omega_0 \wedge \bar{\Omega}_0 \right), \tag{2.5}$$

where Ω_0 is the holomorphic 3-form on X, depending holomorphically on the complex structure moduli. It is convenient to introduce also the normalized 3-form

$$\Omega = e^{\mathcal{K}/2}\,\Omega_0\,. \tag{2.6}$$

The "central charge" of $\Gamma \in H^3(X,\mathbb{Z})$ is given by

$$Z(\Gamma) \equiv \int_X \Gamma \wedge \Omega \equiv \int_\Gamma \Omega\,, \tag{2.7}$$

where we denoted, by slight abuse of notation, the cycle Poincaré dual to Γ by the same symbol Γ.

In the following we will frequently make use of the (antisymmetric, topological, moduli independent) intersection product:

$$\langle \Gamma_1, \Gamma_2 \rangle = \int_X \Gamma_1 \wedge \Gamma_2 = \#(\Gamma_1 \cap \Gamma_2)\,. \tag{2.8}$$

With this notation, we have for a symplectic basis $\{\alpha^I, \beta_I\}$ by definition $\langle \alpha^I, \beta_J \rangle = \delta^I_J$. We will also often use the (symmetric, positive definite, moduli dependent) Hodge product:

$$\langle \Gamma_1, \widehat{\Gamma_2} \rangle = \langle \Gamma_1, *_X \Gamma_2 \rangle = \int_X \Gamma_1 \wedge *_X \Gamma_2\,. \tag{2.9}$$

When the Γ_i denote cohomology classes, their harmonic representative will always be assumed in (2.9).

Every harmonic 3-form Γ on X can be decomposed according to $H^3(X,\mathbb{C}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$ as (for real Γ):

$$\Gamma = i\bar{Z}(\Gamma)\,\Omega - ig^{a\bar{b}}\bar{D}_{\bar{b}}Z(\Gamma)\,D_a\Omega + \text{c.c.}\,, \tag{2.10}$$

where we introduced the Kähler covariant derivative on Z and Ω:

$$D_a \equiv \partial_a + \frac{1}{2}\partial_a\mathcal{K}\,. \tag{2.11}$$

This decomposition is orthogonal with respect to the intersection product (2.8), and diagonalizes the Hodge star operator:

$$*_X\Omega = -i\,\Omega \qquad \text{and} \qquad *_X\, D_a\Omega = i\,D_a\Omega\,. \tag{2.12}$$

For further reference, we write down the following useful identities:

$$\int_X \Omega\wedge\bar{\Omega} = -i\,, \tag{2.13}$$

$$\int_X D_a\Omega\wedge\bar{D}_{\bar{b}}\bar{\Omega} = i\,g_{a\bar{b}}\,, \tag{2.14}$$

$$(d + i\,Q + i\,d\alpha)\left(e^{-i\alpha}\Omega\right) = e^{-i\alpha}\,D_a\Omega\,dz^a\,, \tag{2.15}$$

where α is an arbitrary real function and Q is the chiral connection:

$$Q = \operatorname{Im}\left(\partial_a\mathcal{K}dz^a\right). \tag{2.16}$$

As an example of an application, one can easily check the following expressions for intersection and Hodge products:

$$\langle\Gamma_1,\Gamma_2\rangle = 2\,\operatorname{Im}\Big[-Z(\Gamma_1)\,\bar{Z}(\Gamma_2) + g^{a\bar{b}}\,D_aZ(\Gamma_1)\,\bar{D}_{\bar{b}}\bar{Z}(\Gamma_2)\Big]\,, \tag{2.17}$$

$$\langle\Gamma_1,\widehat{\Gamma_2}\rangle = 2\,\operatorname{Re}\Big[Z(\Gamma_1)\,\bar{Z}(\Gamma_2) + g^{a\bar{b}}\,D_aZ(\Gamma_1)\,\bar{D}_{\bar{b}}\bar{Z}(\Gamma_2)\Big]\,. \tag{2.18}$$

3. The attractor flow equations revisited

We now turn to the investigation of 4d supergravity BPS solutions with charged sources corresponding to D3-branes wrapped around a non-trivial supersymmetric (i.e. special lagrangian) 3-cycle [5] Γ of X. In the mirror IIA picture this corresponds to BPS states with (mixed) 0-, 2-, 4- and 6-brane charge.

Such $\mathcal{N} = 2$ supergravity solutions and the remarkable attractor mechanism emerging in this context were first studied, from supersymmetry considerations, in [1]. An approach based on the bosonic action, which we will also follow here, was pioneered in [2]. Further explorations were made in [8], and various solutions analyzed in [9]–[12]. A rich connection with D-branes, geometry and arithmetic was discovered in [3]. Some recent work on analogous phenomena in five-dimensional theories includes [13].

Part of this section is a review of well known results, though the geometric, manifestly duality invariant setup we use may give a clarifying alternative point of view on some of these. Also, the strategy outlined here to obtain the BPS equations directly from the bosonic action will enable us in section 7 to do the same for the general stationary case (possibly non static, with multiple centers having mutually non-local charges), adding further insight to the solutions of [11]. Furthermore, some subtleties in the derivation of the flow equations will turn out to be relevant for a proper treatment of the solution for conifold charges later on, and finally, an interpretation of the reduced action as that of an effective stretched string will allow us to make contact with the $3-1-7$ brane description of BPS states in $\mathcal{N}=2$ QFT.

So we believe it's worthwhile to revisit this derivation. However, the reader only interested in the resulting equations can safely skip the derivation and proceed with section 4.

3.1 Duality invariant formalism

The relevant bosonic part of the usual 4d low energy effective $\mathcal{N}=2$ supergravity action is, in 4d Planck units:

$$S_{4D} = \frac{1}{16\pi}\int_{M_4} d^4x\sqrt{-G}R - 2g_{a\bar{b}}\,dz^a \wedge *d\bar{z}^{\bar{b}} - \frac{1}{4\gamma^2}\int_{M_4} F^I \wedge G_I\,, \tag{3.1}$$

where γ is a convention dependent number, $F^I = dA^I$ and the G_I are obtained from the F^I using the selfduality constraint (2.2). On the other hand, the bosonic 4d spacetime part of the low energy effective action of a probe D3-brane wrapped around a supersymmetric 3-cycle in the homology class Γ, in a given background, is [5, 6, 7]:

$$S_\Gamma = -\int |Z(\Gamma)|ds + \frac{\sqrt{\pi}}{\gamma}\int \langle \Gamma, \mathcal{A}\rangle\,, \tag{3.2}$$

with $Z(\Gamma)$ as in (2.7), $d\mathcal{A}=\mathcal{F}$, and $\langle\cdot,\cdot\rangle$ denoting the intersection product (2.8). The integral is over the effective particle worldline.

Combining (3.1) and (3.2), assuming Γ to be electric with respect to the choice of symplectic basis (that is, Γ is a linear combination of the α^I), we see that an electromagnetic field produced by such a source with charge Γ satisfies, for any spatial surface S surrounding the source:

$$\int_S \mathcal{F} = \sqrt{4\pi}\gamma\,\Gamma\,. \tag{3.3}$$

Now while the action (3.1) makes four-dimensional general covariance manifest, it is not invariant under electromagnetic duality rotations (i.e. change of symplectic basis in (2.1)). A straightforward, manifestly covariant action exhibiting manifest duality invariance does not exist. In fact, since the 4D theory descends directly from type-IIB supergravity, this problem is equivalent to the nonexistence of a straightforward generally covariant action for the self-dual four-form potential. However, a perfectly satisfactory, though not manifestly covariant action for self-dual forms has been known for quite a while [14], and this action (dimensionally reduced) will actually turn out to be very convenient for our purposes.

To write down this action for an arbitrary background metric, one introduces the usual shift and lapse vectors [15] N_0 and N^i, putting the four-dimensional metric in the form:

$$ds^2 = -N_0^2 dt^2 + G_{ij} \left(dx^i + N^i dt\right)^2 . \tag{3.4}$$

The shift vector determines a three-dimensional 1-form $\mathbf{N} = G_{ij} N^j dx^i$. We will use boldface notation to refer to three-dimensional quantities throughout. Thus $\mathbf{d} = dx^i \, \partial_i$, the 3d Hodge dual (based on G_{ij}) is denoted by $*$, and the spatial part of the total electromagnetic field $\mathcal{F}$ is

$$\boldsymbol{\mathcal{F}} = \mathcal{F}_{ij} \, dx^i \wedge dx^j = \mathbf{d} \, \boldsymbol{\mathcal{A}} . \tag{3.5}$$

The $H^3(X, \mathbb{R})$-valued 3-vector potential $\boldsymbol{\mathcal{A}}$ is considered to be the fundamental variable (instead of the 4-vector potentials A^I in the formulation based on (3.1)). The action obtained from [14] with our compactification assumptions is then:

$$S_{\text{e.m.}} = \frac{1}{4\gamma^2} \int dt \int_{M_3} \int_X \boldsymbol{\mathcal{F}} \wedge \partial_t \boldsymbol{\mathcal{A}} - \left(N_0 \, \boldsymbol{\mathcal{F}} \wedge * \widehat{\boldsymbol{\mathcal{F}}} + \mathbf{N} \wedge * \boldsymbol{\mathcal{F}} \wedge * \boldsymbol{\mathcal{F}} \right) . \tag{3.6}$$

The integral over X yields simply the intersection product (2.8). Since the above expression does not refer to any choice of symplectic basis, it is indeed manifestly duality invariant. The equation of motion following from this action is the self-duality condition (2.2), with $\mathcal{F} = d\mathcal{A}$, where $\mathcal{A}_0$ arises as an integration constant.

Of course, since $\mathcal{A}_0$ does not exist off shell in this formulation, we can no longer use the coupling of the electromagnetic field to sources as in (3.2). Instead, its coupling to charges is implemented by imposing the constraint (3.3), which only involves the spatial fields. Again, no reference to a choice of basis is made. Note that the presence of charges will induce Dirac string singularities in $\boldsymbol{\mathcal{A}}$, or require the introduction of a non-trivial bundle structure.

The coupling of the source to gravity and the scalars remains unchanged.

We will use (3.6) instead of the electromagnetic part of (3.1). In section 7 the full form of this action at non-zero $\mathbf{N}$ will be used to derive the BPS equations for the general stationary case, but it's instructive (and sufficient for most of our purposes) to first consider some simpler cases.

3.2 Reduced action for static spherically symmetric configurations

In [16] it was argued that time independent BPS configurations require a metric that can be expressed in the form

$$ds^2 = -e^{2U}\left(dt + \omega_i dx^i\right)^2 + e^{-2U} dx^i dx^i \,. \tag{3.7}$$

This is the metric ansatz we will use throughout this paper. Usually we will also restrict to asymptotically flat spacetimes, i.e. $U, \omega \to 0$ at spatial infinity. Let us first consider static, spherically symmetric configurations. Then $\omega = 0$ and U is a function of the radial coordinate $r = |\mathbf{x}|$ only. Similarly we take the moduli z^a to be function of r only, and we can assume $\boldsymbol{\mathcal{F}}$ to be of the form

$$\boldsymbol{\mathcal{F}} = \frac{\gamma}{\sqrt{4\pi}} \sin\theta \, d\theta \wedge d\phi \otimes \Gamma \,, \tag{3.8}$$

where θ and ϕ are the usual angular coordinates and $\Gamma \in H^3(X, \mathbb{Z})$ is the charge of the source. Then the total electromagnetic field is, with $\tau \equiv 1/r$:

$$\mathcal{F} = \boldsymbol{\mathcal{F}} + *_4 \widehat{\boldsymbol{\mathcal{F}}} = \frac{\gamma}{\sqrt{4\pi}} \left(\sin\theta \, d\theta \wedge d\phi \otimes \Gamma + e^{2U} d\tau \wedge dt \otimes \widehat{\Gamma} \right), \tag{3.9}$$

which trivially satisfies the required equations of motion and duality constraints $d\mathcal{F} = 0$ and $\mathcal{F} = *_4 \widehat{\mathcal{F}}$.

In terms of the inverse radial coordinate $\tau = 1/r$, the total action per unit time reduces, under these assumptions, and dropping a total derivative proportional to $\ddot{U}$, simply to:

$$S_{\text{eff}} = \frac{S}{\Delta t} = -\frac{1}{2} \int_0^\infty d\tau \left\{ \dot{U}^2 + g_{a\bar{b}} \, \dot{z}^a \dot{\bar{z}}^{\bar{b}} + e^{2U} V(z) \right\} - \left(e^U |Z| \right)_{\tau=\infty} , \tag{3.10}$$

where the dot denotes derivation with respect to τ and (cf. (2.18))

$$V(z) = \frac{1}{2} \langle \Gamma, \widehat{\Gamma} \rangle \tag{3.11}$$

$$= |Z|^2 + g^{a\bar{b}} \, D_a Z \, \bar{D}_{\bar{b}} \bar{Z} = |Z|^2 + 4 g^{a\bar{b}} \, \partial_a |Z| \, \bar{\partial}_{\bar{b}} |Z| \tag{3.12}$$

with $Z = Z(\Gamma)$. The "potential" $e^{2U} V(z)$ is proportional to the electromagnetic energy density. The boundary term in (3.10) comes from (3.2). In principle, this reduced action has to be supplemented by the constraints coming from variations of the metric (consistent with spherical symmetry) other than the U mode. In particular here this gives the constraint $\dot{U}^2 + \|\dot{z}\|^2 - e^{2U} V(z) = 0$. However, as we will see, this turns out to follow already from (3.10) (with the given source coupling and allowing arbitrary variations of the fields at $\tau = \infty$ in the variational principle). So we will simply proceed with the analysis of the action as it stands.

Note that (minus) this effective action per unit time can be interpreted as describing a non-relativistic particle moving in (U, z)-space, subject to the potential $-e^{2U} V(z)$, with time τ [2] (figure 1). Only the initial ($\tau = 1/r = 0$) position of

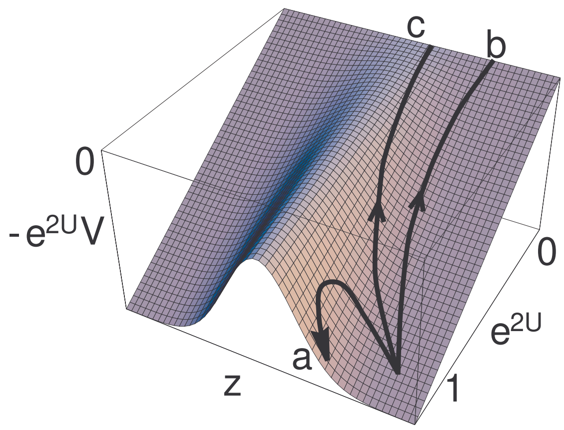

Figure 1: Typical case sketch of the potential $-e^{2U}V$ in which the effective particle is moving, plotted as a function of e^{2U} and the moduli z. Of the three plotted trajectories, only (c) satisfies the required asymptotic conditions yielding a BPS black hole.

this effective particle is fixed. The $\tau \to \infty$ asymptotic behavior is given by requiring vanishing of the boundary terms at $\tau = \infty$ when varying the fields. This yields for $\tau \to \infty$

$$\dot{U} \longrightarrow -e^U |Z|\,, \tag{3.13}$$
$$\dot{z}^a \longrightarrow -2e^U g^{a\bar{b}}\, \bar{\partial}_{\bar{b}} |Z|\,. \tag{3.14}$$

Incidentally, these are precisely the attractor flow equations, as we will see below. Note that this condition also implies the vanishing of the (conserved) effective particle's energy, $E_{\text{eff}} = \dot{U}^2 + \|\dot{z}\|^2 - e^{2U}V(z) = 0$, which is precisely the "additional" constraint discussed earlier.

3.3 Interpretation as a static string action

In fact, the reduced action (3.11) can also be interpreted — and this is perhaps more natural — as a Polyakov action for a static string in (U, z) space, with (variable) tension $T = e^U \sqrt{V(z)}$, in the background target space metric

$$ds^2 = -dt^2 + dU^2 + g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}\,, \tag{3.15}$$

and worldsheet metric $\sim \text{diag}(-T, T^{-1})$ with respect to the worldsheet time resp. space coordinates t and τ. The vanishing of the effective particle's energy E_{eff} is equivalent to the Virasoro constraint, which can be used to transform the action to the Nambu-Goto form

$$S = -\int dt\, d\tau\, e^U \sqrt{V} \sqrt{\dot{U}^2 + \|\dot{z}\|^2}\,. \tag{3.16}$$

The asymptotic condition (3.13)–(3.14), being equivalent to the attractor flow equations, forces the endpoint of the string at $\tau = \infty$ to be fixed at an attractor point (see below). The other endpoint is fixed at the values of the moduli and U at spatial infinity in the supergravity picture. The equations of motion determine the string to be a geodesic in (U, z) space.

This interpretation, in a suitable rigid (i.e. gravity decoupling) limit of $\mathcal{N} = 2$ supergravity [7, 17, 12] (leading to Seiberg-Witten theory [18][1] and its generalizations), makes direct contact with the description [20] of BPS states in $\mathcal{N} = 2$ quantum field theories as stretched strings in a non-trivial background. For example, repeating the above analysis for a charge (n_e, n_m) state in the pure SU (2) Seiberg-Witten effective theory (or considering a suitable rigid limit of supergravity), one finds an effective reduced action with similar structure, with one modulus u, $V(u) = |n_e + n_m \tau|^2 / \operatorname{Im} \tau$ (where $\tau(u)$ is the usual modular parameter of the SW Riemann surface), and evidently $U \equiv 0$. For, say, a magnetic monopole, the attractor point turns out to be its vanishing mass point $u = 1$ (see also section 5.2). Hence our effective string will be a geodesic stretched in the Seiberg-Witten plane, between an arbitrary modulus $u_{\tau=0}$ and $u = 1$. Thus we arrive precisely at the picture of [20].

We will return to this point later on. In particular, the phenomenon of three-pronged strings [21] appearing in this context will turn out to be related to the resolution of some intriguing paradoxes.

3.4 BPS equations of motion

The BPS equations of motion can be obtained from (3.10) by the usual Bogomol'nyi trick of completing squares. This can be done in two ways, yielding two different forms of the BPS equations. The first way is well known [2]:

$$\begin{aligned} S_{\text{eff}} = & -\frac{1}{2} \int_0^\infty d\tau \left\{ (\dot{U} \pm e^U |Z|)^2 + \left\| \dot{z}^a \pm 2 e^U g^{a\bar{b}} \bar{\partial}_{\bar{b}} |Z| \right\|^2 \right\} \pm \\ & \pm e^U |Z| \Big|_{\tau=0}^{\tau=\infty} - \left(e^U |Z| \right)_{\tau=\infty} \end{aligned} \tag{3.17}$$

leading to the BPS equations

$$\dot{U} = -e^U |Z| \,, \tag{3.18}$$

$$\dot{z}^a = -2 e^U g^{a\bar{b}}\, \bar{\partial}_{\bar{b}} |Z| \,. \tag{3.19}$$

This is the form of the equations found in [2]. The other sign possibility is excluded by the asymptotic condition (3.13)–(3.14), or alternatively, by requiring physical acceptability: "wrong sign" solutions have runaway behavior, severe curvature singularities at finite distance, negative ADM mass, and are gravitationally repulsive.

[1] For an analysis of BPS solutions of pure low energy effective $\mathcal{N} = 2$ SU (2) Yang-Mills theory, see [19]. The solutions obtained there can be seen to be the rigid limit of supergravity attractor flows [12].

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The second way of squaring the action uses the Hodge product (2.9); if for a 3-form C on X we write

$$|C|^2 = \langle C, \widehat{C}\,\rangle \tag{3.20}$$

and we denote the position dependent phase of the central charge as

$$\alpha = \arg Z(\Gamma) \tag{3.21}$$

then we have, using (2.13)–(2.15):

$$\begin{aligned} S_{\text{eff}} = &-\frac{1}{4}\int_0^\infty d\tau\, e^{2U}\Big|2\ \text{Im}\left[(\partial_\tau + i\,Q_\tau + i\,\dot\alpha)\left(e^{-U}e^{-i\alpha}\Omega\right)\right] + \Gamma\Big|^2 - \\ &-\left(e^U|Z|\right)_{\tau=0}, \end{aligned} \tag{3.22}$$

where $Q_\tau = \text{Im}(\partial_a\mathcal{K}\dot z^a)$, as in (2.16). (We have left $U_{\tau=0}$ arbitrary here, though in the asymptotically flat case this is zero of course.) Note that we take the holomorphic 3-form Ω and the unnormalized holomorphic Ω_0 (cf. (2.6)) to be only dependent on the spacetime coordinates through the moduli $z^a(\tau)$. This is in contrast to ref. [9], where a (convenient) explicit position dependence of normalisation and phase of Ω_0 was chosen.[2] However, from a Calabi-Yau geometrical point of view it is perhaps more natural to pick a dependence only through the moduli. In numerical computations, this has the further advantage that one can work with a fixed expression for Ω. Furthermore, in this way the phase α appears naturally in the equations, and this phase (which can be identified with the phase of the conserved supersymmetry [3]) will play a crucial role in the comparison with geometrical results on special lagrangian manifolds [22].

The BPS equation following from (3.22) is again obtained by putting the square to zero:

$$2\ \text{Im}\left[(\partial_\tau + i\,Q_\tau + i\,\dot\alpha)\left(e^{-U}e^{-i\alpha}\Omega\right)\right] = -\Gamma\,. \tag{3.23}$$

However, by taking the intersection product with Γ on both sides of the equation, and using (3.21), it follows that $Q_\tau + \dot\alpha = 0$, hence the BPS equation becomes simply

$$2\,\partial_\tau\ \text{Im}\left(e^{-U}e^{-i\alpha}\Omega\right) = -\Gamma\,. \tag{3.24}$$

Conversely, by taking the intersection product of the latter equation with $\bar\Omega$, using (2.15) and (2.13), and taking the imaginary part of the result, one obtains again $Q_\tau + \dot\alpha = 0$, and hence (3.23). Now (3.24) readily integrates to

$$2\ \text{Im}\left(e^{-U}e^{-i\alpha}\Omega\right) = -\Gamma\,\tau + 2\ \text{Im}\left(e^{-U}e^{-i\alpha}\Omega\right)_{\tau=0}. \tag{3.25}$$

This is a powerful result, as it *solves*, in principle, the BPS equations of motion of the system. To bring it in a perhaps more familiar form, choose a symplectic basis $\{\alpha^I, \beta_I\}$ of $H^3(X,\mathbb{Z})$, write $\Gamma = -q_I\alpha^I + p^I\beta^I$, define the holomorphic periods

[2] One is free to make such a gauge choice, at least locally, since phase and normalisation of Ω_0 do not enter the action (3.1).

$X^I = \langle \alpha^I, \Omega_0 \rangle$, $F_I = \langle \beta_I, \Omega_0 \rangle$, and take intersection products of this basis with the above equation. This gives:

$$2\, e^{-U+\mathcal{K}/2}\, \mathrm{Im}\left(e^{-i\alpha} X^I\right) = \frac{p^I}{r} + c^I\,, \tag{3.26}$$

$$2\, e^{-U+\mathcal{K}/2}\, \mathrm{Im}\left(e^{-i\alpha} F_I\right) = \frac{q_I}{r} + d_I\,, \tag{3.27}$$

where we re-introduced $r = 1/\tau$, and c^I, d_I are constants. If, as in [8, 9], we would pick an Ω_0-gauge where $\mathcal{K} \equiv 2U$ and $\alpha \equiv 0$, one retrieves the expressions appearing in those references. Note also that the flow equations (3.18)–(3.19) are nothing but the projections of (3.23) on $e^{-i\alpha}\Omega$ resp. $e^{-i\alpha} D_a \Omega$.

Of course, finding the explicit flows in moduli space from (3.25) requires inversion of the periods to the moduli, which in general is not feasible analytically. However, in large complex structure approximations [9] or numerically for e.g. the quintic, this turns out to be possible.

One final remark is in order. The solution (3.25) was derived from the action (3.22) under the implicit assumption that the quantity between brackets is not proportional to (the Poincaré dual of) a vanishing cycle ν (that is, a cycle for which the Hodge square $\langle \nu, \widehat{\nu} \rangle = 0$, like the conifold cycle at a conifold point of moduli space). If that is the case, the Hodge square appearing in (3.22) is automatically zero, no matter what the expression inside the $|\cdot|$ is (as long as it is finitely proportional to a vanishing cycle). Actually, a we will see, such situations *do* occur naturally, and the previous remark should eliminate possible confusion there.

4. Attractors and existence of BPS states

4.1 Properties of some solutions

In what follows we will assume asymptotic flatness, i.e. $U_{\tau=0} = 0$. Then solutions to (3.18)–(3.19) saturate the BPS bound

$$M = |Z_{\tau=0}|\,. \tag{4.1}$$

All mass is located in the fields: the "bare mass" contribution $(e^U|Z|)_{\tau=\infty}$ is zero. Indeed, (3.18) and (3.19) imply that both e^U and $|Z|$ are monotonously decreasing functions satisfying the estimate $e^U|Z| \leq |Z|/(1+|Z_\infty|\tau)$, hence $e^U|Z| \to 0$ when $\tau \to \infty$. More precisely, equation (3.19) implies

$$\partial_\tau |Z| = -4 e^U g^{a\bar{b}}\, \partial_a |Z|\, \bar{\partial}_{\bar{b}} |Z| \leq 0\,, \tag{4.2}$$

so the flows in moduli space described by (3.18) and (3.19) converge to minima of the central charge modulus $|Z(\Gamma)|$ (figure 2). Therefore the moduli values at the horizon are generically invariant under continuous deformations of the moduli at spatial infinity, and hence only dependent on the charge Γ, a phenomenon referred to as the

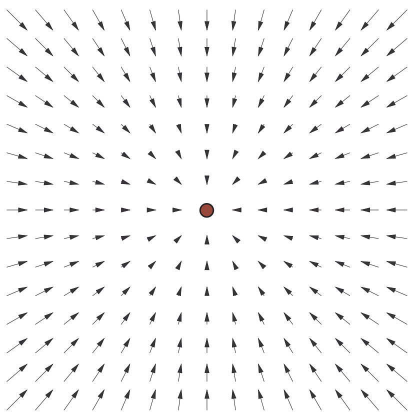

Figure 2: Typical flow gradient field in moduli space (represented here by the z-plane) close to a generic attractor point with non-zero minimal $|Z|$. The gradient vectors vanish at the attractor point.

attractor mechanism. The attractor values of the moduli correspond to Calabi-Yau manifolds with very remarkable arithmetic properties, as explored in detail in [3].

4.1.1 Black holes

The above estimate also implies that when $Z_\infty \neq 0$, the solution is a black hole with horizon at $\tau = \infty$. From the form of the metric (3.7) and direct analysis of equation (3.18) in the limit $\tau \to \infty$, the near horizon geometry can be seen to be $AdS_2 \times S^2$:

$$ds^2_{NH} = -\frac{r^2}{|Z_\infty|^2} dt^2 + \frac{|Z_\infty|^2}{r^2} d\mathbf{x}^2 \,. \tag{4.3}$$

The corresponding macroscopic entropy is $A/4 = \pi |Z_\infty|^2$. These black holes have been studied extensively in the literature [1, 2], [8]–[11].

4.1.2 Empty holes

When the D3-brane wraps a conifold cycle, i.e. a cycle vanishing at a conifold point, the minimal value of the central charge modulus is zero, and the above discussion of the generic case does no longer apply. However, since conifold cycles are known to exist (close to a conifold point) as special lagrangian submanifolds, and therefore as physical BPS particles, it is natural to ask what the corresponding supergravity solution looks like.

Again, the flow in moduli space will converge to a point where $|Z|$ is minimal, which in this case is a point on the conifold locus, where Z vanishes. At the conifold locus, the Calabi-Yau degenerates and we get an additional massless hypermultiplet in the low energy theory, so there we cannot necessarily trust our supergravity approximation. However, the results obtained are physically pleasing and interesting, so we will ignore this potential problem and proceed.

For simplicity, following [3], we will consider only one modulus, z, which we define to be the holomorphic (Ω_0) period of the vanishing cycle. Then the Kähler potential and metric close to the conifold point ($z \to 0$) can be taken to be:

$$e^{-\mathcal{K}} \approx k_1^2 + \frac{1}{2\pi}|z|^2 \ln |z|^2 + k_2 \operatorname{Re} z \,, \tag{4.4}$$

$$g_{z\bar{z}} \approx \frac{1}{2\pi k_1^2} \ln |z|^{-2} \,, \tag{4.5}$$

where k_1 and k_2 are positive constants. This geometry can be observed for instance close to the conifold point of the quintic, or (in a rigid limit) close to the massless monopole or dyon singularities in Seiberg-Witten theory.

The central charge of N times the vanishing cycle close to $z = 0$ is $Z = \frac{N}{k_1} z$ and the attractor flow equations in this limit are

$$\dot{U} = -\frac{N}{k_1} e^U |z| \,, \tag{4.6}$$

$$\dot{z} = -2\pi k_1 N e^U \frac{z}{|z|} \frac{1}{\ln |z|^{-2}} \,, \tag{4.7}$$

with solution (approximately for $z \to 0$) given by:

$$\arg z = \text{const.} \,, \tag{4.8}$$

$$|z| \ln |z|^{-1} = \begin{cases} \pi k_1 N \, e^{U_*} \, (\tau_* - \tau) & \text{for } \tau < \tau_* \\ 0 & \text{for } \tau \geq \tau_* \,, \end{cases} \tag{4.9}$$

$$U = \frac{1}{4\pi k_1^2} |z|^2 \ln |z|^{-2} + U_* \,, \tag{4.10}$$

where τ_* and U_* are constants depending on initial conditions.

So here the attractor point $z = 0$ is reached at *finite non-zero* coordinate distance $r_* = 1/\tau_*$ from the origin. In the core region $r < r_*$, the fields z and U are constant and the geometry is flat. There is no horizon and the core contains no energy ($\dot{U} = \dot{z} = 0$ and $V(z) = 0$), hence the name "empty hole" (figure 3). The solution $z(\tau)$ is once and $U(\tau)$ twice continuously differentiable at $\tau = \tau_*$, so the curvature stays finite (and can be made arbitrary small by taking N sufficiently large). However, higher derivatives diverge at $\tau = \tau_*$, so strictly speaking the (two derivative) supergravity approximation breaks down here.[3] Nevertheless, we believe this is a physically sensible solution. For instance, one could use it (in the straightforward multicenter extension given section 4.2) to compute the dynamics of a large number of slowly moving empty holes, in moduli space approximation, with sensible results. Note that due to the fact that these solutions are *not* black holes, one will obtain a moduli space geometry for nearly coincident centers which is completely different

[3] This can perhaps be cured by including higher derivative terms or the new massless hypermultiplet in the effective action, smoothing out the solution, but presumably not changing it too much.

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Figure 3: Energy density sketch of an "empty hole".

from the black hole one discussed in [23, 24]. In particular, there will presumably be no coalescence, in agreement with the physical expectation that no BPS *bound* states should exist for branes wrapping a conifold cycle [25].

We could also have used (3.25) to construct this solution (in particular cases this would in fact be a more powerful method to extract exact results, also away from the near conifold limit). However, naive application of this formula would lead to a field configuaration that is not constant inside the core: at $\tau = \tau_*$, the central charge phase α jumps discontinuously from α_* to $\alpha_* + \pi$, and for $\tau > \tau_*$, one gets the "solution" corresponding to the flow equations (3.18)–(3.19) with the opposite (=wrong) sign. As discussed in section 3.4, this is not an acceptable solution; it is not BPS, and physically ill-behaved.

The way out of this paradox is the remark given at the end of section 3.4: equation (3.25) needs only to be satisfied down to the radius where the conifold attractor point is reached. If we keep the fields constant below this radius, the BPS condition is automatically satisfied. This eliminates some confusion arising in the literature in this context.

Note that even though the solution (4.8)–(4.10) was derived in the near conifold limit, the conclusion that the attractor point is reached at finite τ is also true for moduli at infinity farther away from the conifold point, since the region where the approximation becomes valid will in any case be reached after finite τ. Furthermore solutions at different N are related by simple scaling; the core radius is proportional to N. So the solution will never be a black hole, no matter where we start in the moduli space, and no matter how many particles we put on top of each other. The attractor mechanism causes the mass to stay outside the Schwarzschild radius, protecting the configuration from gravitational collapse.

If the modulus z_0 at spatial infinity $\tau = 0$ is sufficiently small, the core radius is given by $r_* = \frac{\pi k_1 N}{|z_0| \ln |z_0|^{-1}}$. In the zero mass limit $z_0 \to 0$, the core radius goes to infinity, leaving a completely flat space. If on the other hand one boosts up the particle while sending $z_0 \to 0$, in such way that the total energy remains constant, one obtains [12] in the limit $z_0 \to 0$ the Aichelburg-Sexl shockwave metric [26] for a massless particle moving at the speed of light. Again, this is physically sensible.

Finally note that we have derived the empty hole solution assuming all charge to be located at $\mathbf{x} = 0$. However, exactly the same solution for U and the moduli would have been obtained for any spherically symmetric charge distribution inside the core region. In particular the energy density and space curvature would have been the same. In that sense the charge is actually *delocalized*. It could for example be a spherical shell of radius r_* (this is perhaps the most natural location of the charges, as the "emptyness" of the core then becomes quite intuitive).

All this is of course very reminiscent of the enhançon mechanism of [27]. One could say that the massless conifold particle is the "enhançon" curing the repulson singularity one would obtain for example by applying naively formula (3.25). The main difference is that there is no enhanced gauge symmetry in the core region, but rather an additional massless charged hypermultiplet.

It would be interesting to find out whether empty holes, like their black hole cousins [28], also have a Maldacena dual [29] QFT description.

4.1.3 No holes

As observed in [3], the flow equations (3.18)–(3.19) do not always have a solution: if the attractor point of the flow happens to be a simple zero of Z, at a regular point of moduli space, the flow will reach $Z = 0$ at finite $\tau = \tau_*$ and *cannot* be continued in a BPS way to the interior region $\tau > \tau_*$ (see also figure 4). The basic difference with the previous case is the absence of the "damping" factor $1/\ln|z|^{-2}$ in the inverse metric on the right-hand side of (3.19) (or (4.7)), so that the constant field configuration at $Z = 0$ is no longer a solution. On the other hand, by taking the charge sufficiently large, the curvature can be made again arbitrary small, so the absence of a supergravity solution should be quite meaningful.

Physically, one indeed does not expect a BPS state with charge Γ to exist in a vacuum near a regular point where $Z(\Gamma) = 0$: such a particle would be massless at $Z = 0$, which (by integrating it out) should lead to a singularity in moduli space [25], in contradiction with the supposed regularity of the point under consideration.

4.2 Equal charge multicenter solutions

The single center configuration discussed above is readily extended to the multicenter case with equal (or parallel) charges in the centers. (Multicenter solutions with non-parallel charges are considerably more involved, and will be discussed in section 7.) One simply replaces $\tau = 1/|\mathbf{x}|$ by

$$\tau \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|\mathbf{x} - \mathbf{x}_i|}\,, \tag{4.11}$$

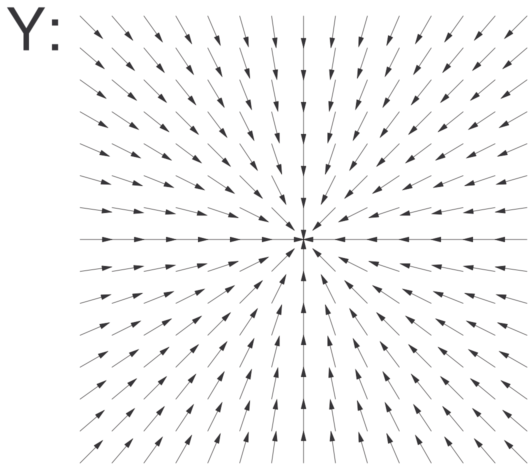

Figure 4: Flow gradient field in the $Y = \int_\Gamma \Omega_0$-plane close to a regular point $Y = 0$ in moduli space where the central charge vanishes. The gradient vectors do not vanish at the attractor point, leading to a breakdown of the flow.

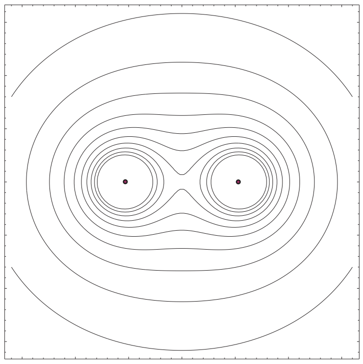

Figure 5: Some surfaces of equal τ in the 2-center case.

where the $\mathbf{x}_i$ denote the positions of the particles, each with charge Γ (figure 5). Since the complete setup is formally the same as for the spherically symmetric case, so are the attractor flow equations. Therefore, everything said about the spherically symmetric case applies to the multicenter case as well.

For nearly coincident centers, the black hole near horizon geometry now becomes "fragmented" $AdS_2 \times S^2$, as discussed in [28].

Though a proof for the general case is still lacking, it is expected [30] that the moduli space geometry for the dynamics of slowly moving centers can be derived [24] from the potential

$$L = \int d^3\mathbf{x}\, e^{-4U} \,. \tag{4.12}$$

4.3 Existence of BPS states

The issue of existence of BPS states with given charge in theories with $\mathcal{N}=2$ supersymmetry in four dimensions is non trivial and profound. The simplest example of such a theory is probably SU (2) $\mathcal{N}=2$ Yang-Mills theory. The low energy dynamics of this theory was exactly solved in [18], where it was found that the BPS spectrum at weak coupling consists of the gauge boson and a tower of dyons of arbitrary integer electric charge and one unit of magnetic charge, while at strong coupling it consists solely of the magnetic monopole and the "elementary" dyon with one unit of electric and one unit of magnetic charge. Here "strong coupling" has the precise meaning of being inside a certain curve in the one-dimensional moduli space, called the curve of marginal stability. At this curve, the various BPS particles have parallel central charges, so that they become only marginally stable against decay into constituents.

Similar phenomena are expected for type-II string theory compactified on a Calabi-Yau 3-fold. Here the subject is intimately related to the existence of D-branes wrapped on supersymmetric cycles, since these are the objects that represent the BPS states. For instance in type-IIB theory, at least in the large complex structure limit, existence of a BPS state of charge $\Gamma \in H^3(X,\mathbb{Z})$ is equivalent with existence of a special lagrangian submanifold in the homology class (Poincaré dual to) Γ [5]. In type-IIA theory, in the large volume limit, it is equivalent with existence of holomorphic submanifolds endowed with certain holomorphic bundles (or more precisely sheaves). At certain special points in moduli space, existence can be proven using the boundary state formalism. Recently, the problem has been studied intensively from various points of view: special lagrangian submanifolds [5, 31, 22, 32, 33], holomorphic geometry and boundary states [34]–[41] and low energy effective supergravity [3, 36].

We will study this problem from the latter point of view, namely the low energy supergravity theory. The idea [3] is as follows. If a certain charge supports a BPS state, one certainly would expect a corresponding 4d supergravity solution to exist, at least for sufficiently large charge, such that the supergravity approximation can be trusted. The converse statement is perhaps less evident, but with the knowledge that some charges indeed do not have BPS supergravity solutions (see section 4.1.3), it is quite tempting to conjecture an exact correspondence, at least for sufficiently large charges. Clearly, considering the degree of difficulty of the problem in other approaches, such a correspondence would be very powerful.

The above considerations were used in [3] to arrive at the following proposal for an existence criterion for BPS states with given charge. Choose moduli ${z^a}_0$ at spatial infinity and a charge Γ, and denote the minimal value of $|Z(\Gamma)|$ where the solution of (3.18)–(3.19) flows to by $|Z|_{\min}$. There are three distinguished cases.

- *Type a*: $|Z|_{\min} \neq 0$. The attractor flow exists and yields a regular BPS black hole solution. In this case one expects to have a BPS state in the theory with the given charge. Note that if the existence of a BPS state in a certain vacuum ${z^a}_0$

is thus established, it will also exist in any other vacuum that lies "upstream" the Γ attractor flow passing through the point $z^a{}_0$, where "upstream" means in the opposite direction of the flow given by (3.18)–(3.19). Since $|Z|$ has no maxima in moduli space [2], the upstream flows will tend to regions of moduli space at infinite distance, like the large complex structure limit. This also explains to a certain extent why BPS states are more likely to exist at large complex structure than in the bulk of modulispace.

- *Type b*: the flow tends to a singularity or a boundary of moduli space, where $|Z|$ might or might not vanish. More information is needed to decide whether the BPS state exists or not.

- *Type c*: $|Z|_{\min} = 0$, and this minimum is reached at a regular point in moduli space. As discussed in section 4.1.3, the flow breaks down and the charge is expected not to support a BPS state.

Though this proposed criterion works nicely for e.g. T^6 [3], it can fail in more general cases, as we will argue in the next section. More precisely, it turns out that some type c cases *do* correspond to BPS states present in the theory.

5. Puzzles and paradoxes

5.1 Puzzle 1: Solution suicide; states at the Gepner point of the quintic.

We start by considering the example of the quintic Calabi-Yau, first analyzed in great detail in [42]. In particular, we will study BPS states in type-IIB theory on the mirror quintic W (or equivalently in type IIA on the quintic M itself). This manifold can be obtained [43] as a $\mathbb{Z}_5^3$ quotient of the manifold in $\mathbb{P}^4$ given by the equation

$$W : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\,\psi\, x_1 x_2 x_3 x_4 x_5 = 0\,. \tag{5.1}$$

The transformation $\psi \to \omega\psi$ with $\omega^5 = 1$ can be undone by a coordinate transformation $x_1 \to \omega^{-1} x_1$, and thus the complex structure moduli space of W can be parametrized by ψ^5. The moduli space has three singularities: the Gepner point $\psi^5 = 0$, which is a $\mathbb{Z}_5$ orbifold singularity, the conifold point $\psi^5 = 1$, where a 3-cycle vanishes, and the large complex structure limit $\psi^5 = \infty$, mirror to the large volume limit of the quintic.

In [35], building on [34], the D-brane spectrum of this theory was studied, mainly from the conformal field theory perspective. In particular the existence of a number of BPS states was established at the Gepner point $\psi = 0$. These states were labeled as $|00000\rangle_B, |10000\rangle_B, \ldots, |11111\rangle_B$. The state $|00000\rangle_B$ corresponds to a D3-brane wrapped around the conifold cycle on the type-IIB side, and to a D6-brane on the

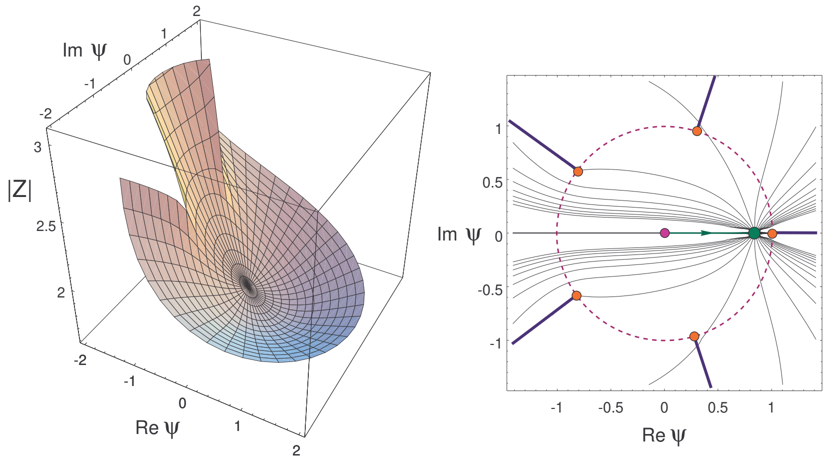

Figure 6: *Left*: modulus of the central charge as a function of ψ for the state $|11000\rangle_B$. The variable ψ parametrizes a five-fold cover of the moduli space around the Gepner point $\psi = 0$. The discontinuities in the graph are due to monodromies around the conifold points $\psi^5 = 1$. There is a regular non-zero minimum $|Z|_{\min} \approx 1.61$ at $\psi \approx 0.85$. *Right*: corresponding flows in the ψ-plane. The five fat blue lines are the cuts for the periods, starting at the conifold points. The green line with arrow from $\psi = 0$ to the attractor point is the flow for the Gepner vacuum.

type-IIA side.[4] It becomes massless at the conifold point $\psi = 1$. The state $|10000\rangle_B$ has two units of D6-brane charge and five units of D2-brane charge in the type-IIA theory, and the state $|11000\rangle_B$ has one unit of D6- and five units of D2-brane charge. The expected dimension of the deformation moduli space of these three states is respectively 0, 4 and 11.

According to the existence criterion of section 4.3, we should find "good" attractor flows with $\psi = 0$ at spatial infinity for all these states; they should not be of type c. To address this question, one needs the exact moduli space geometry and central charges (Ω-periods) at arbitrary points in moduli space for the charges under consideration. From the results of [42, 35, 44], all this is indeed available, in terms of certain Meijer functions [44] of the modulus ψ^5. It is still hard then to tackle this problem analytically, but numerically using for instance Mathematica, it becomes quite tractable.

As an example, we show in figure 6 the modulus of the central charge Z as a function of ψ for the state $|11000\rangle_B$. In this case we find indeed a nice regular BPS black hole solution, with $|Z|_{\min} \approx 1.61$ at the attractor point $\psi \approx 0.85$ (to make the supergravity approximation valid, we should actually put a large number N of these charges on top of each other, but this simply rescales the solution). The same

[4]The identification of the type-IIA D-brane charges depends on the chosen analytic continuation to large complex structure, so it has some intrinsic arbitrariness (see [35] for some discussion of this point).

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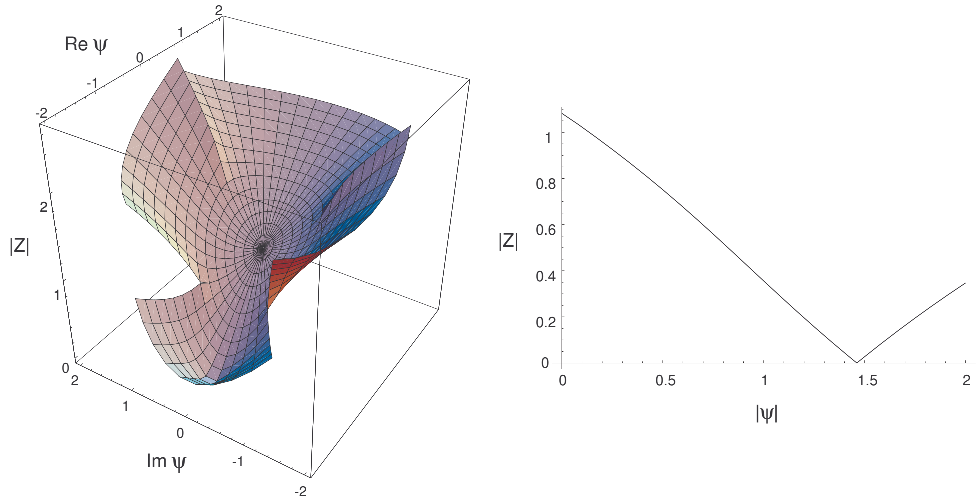

Figure 7: *Left:* modulus of the central charge as a function of ψ for the state $|10000\rangle_B$ (notice the change in viewpoint with respect to figure 6: the ψ-plane is rotated 90 degrees). There is a regular zero at $\psi \approx -1.46$. *Right:* $|Z|$ as a function of $|\psi|$ on the negative real axis.

is true for $|11100\rangle_B$ ($|Z|_{\min} \approx 2.78$ at $\psi \approx -0.51$), for $|11110\rangle_B$ ($|Z|_{\min} \approx 4.58$ at $\psi \approx -0.15$), and for $|11111\rangle_B$ ($|Z|_{\min} \approx 7.43$ at $\psi \approx -0.07$). For $|00000\rangle_B$ we find an empty hole solution with attractor point $\psi = 1$.[5]

However, as also noticed in [36], for the state $|10000\rangle_B$, we are in trouble. As shown in figure 7, the attractor point $\psi \approx -1.46$ is a regular zero of $|Z|$, so we have a type c situation: the supergravity solution does not exist!

Note that, though in conflict with the criterion of section 4.3, this result is *not* in contradiction with the physical expectation that a charge with $Z = 0$ at a regular point cannot support BPS states *in a neighborhood* of that point: the zero $\psi = \psi_* \approx -1.46$ and the Gepner point $\psi = 0$, where the existence of the state is established, can be separated by a line of marginal stability where the state decays into lighter BPS constituents. So it is perfectly possible to have a BPS state with the given charge at $\psi = 0$ and no such state close to $\psi = \psi_*$.

Still, we clearly do have a physical problem here. This can be seen most clearly by considering the following thought experiment (figure 8). Imagine a very large number of particles with the given charge on a huge sphere of radius R, in a vacuum with $\psi = 0$. For $R \to \infty$ we expect to be allowed to neglect the backreaction of the particles on the bulk fields, and the description of these particles as CFT boundary states in the fixed background should be valid. Since we know from [34, 35] that the CFT boundary states corresponding to these particles exist and are BPS

[5] Incidentally, in all cases we find $|Z(\psi)|$ to be symmetric under $\psi \to \bar{\psi}$, illustrating the rather special character of the boundary states constructed in [34, 35].

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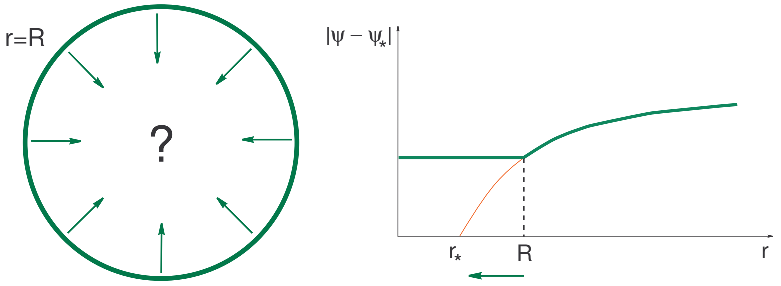

Figure 8: *Left:* shell of BPS particles with zero central charge at the regular point $\psi = \psi_*$, slowly moving inward. *Right:* sketch of $|\psi - \psi_*|$ as a function of the distance r from the origin. The thick green line is the actual value when the radius of the sphere equals R. The thin red line shows how $|\psi - \psi_*|_{r=R}$ would progress when the shell moves further inward.

in the given vacuum, such a configuration should indeed exist and be BPS. Now give each of the particles the same very small inward velocity, and let us approximate the particle cloud as a uniformly charged spherical shell of adiabatically decreasing radius R. When the sphere becomes smaller, the collective backreaction becomes more important: outside the sphere, the fields will be given by the attractor flows (3.18)–(3.19); inside the sphere, the fields are constant. Note that this configuration is indeed BPS: the energy stored in the bulk fields outside the shell is $E_{\text{out}} = |Z|_{r=\infty} - (e^U|Z|)_{r=R}$, the energy of the shell itself is $E_{\text{shell}} = (e^U|Z|)_{r=R}$, and the energy stored in the fields inside the shell is zero, adding up to a total energy $E_{\text{tot}} = |Z|_{r=\infty}$.

But if this motion goes on and nothing happens, we run into disaster: when R becomes smaller than the non-zero radius r_* where the attractor flow breaks down, we no longer have a sensible solution! Moreover, by the physical argument given earlier, we actually expect that the particle cloud does not even exist anymore at this point...

We propose a way out in section 6.

5.2 Puzzle 2: Monodromy murder; dyons in Seiberg-Witten theory

For our second (but closely related) puzzle, we consider the monopole in the Seiberg-Witten low energy effective theory for SU(2) $\mathcal{N} = 2$ Yang-Mills [18]. This theory can be obtained from the $\mathcal{N} = 2$ supergravity theory describing the low energy physics of type-II string theory compactified on a suitable Calabi-Yau manifold, in a certain rigid (=gravity decoupling) limit [45, 17, 12, 7]. The BPS solutions of this effective abelian theory (see [19] for a discussion taking into account non-abelian corrections) can correspondingly be obtained as rigid limits of supergravity attractor flows [12]. Because gravity decouples, U is zero everywhere. The attractor flow equation for the modulus $u(\tau)$ is[6]

$$\dot{u} = -\sqrt{2}\, g^{u\bar{u}}\, \partial_{\bar{u}} |Z| \,, \tag{5.2}$$

where $g^{u\bar{u}}$ is the inverse Seiberg-Witten metric and Z is the central charge; for electric charge n_e and magnetic charge n_m, this is $Z = n_e a(u) + n_m a_D(u)$, where a and a_D are given by certain hypergeometric functions [18].

Because $Z(u)$ is now analytic, the only possible minima of $|Z|$ are zeros. In fact, it is easy to see from (5.2) that the flows are lines of constant Z-phase, which of course necessarily end on a zero of Z. As before, to have a solution, the zero cannot be at a regular point, so it should be at the singularity $u = 1$ where the monopole becomes massless, or at $u = -1$ where the elementary dyon becomes massless. Therefore, the only solutions to (5.2) are of the empty hole type: the monopole, with attractor point $u = 1$, and the elementary dyon, with attractor point $u = -1$, plus of course their oppositely charged partners. In a neighborhood of their respective attractor points, with the choice of period cuts shown in figure 9, the monopole has charge $(n_e, n_m) = (0, 1)$, while the elementary dyon gets assigned the charge $(1, 1)$ above the cut and $(1, -1)$ below.

Again, we are facing a puzzle. It is well known that at weak coupling (that is, outside the line of marginal stability given by $a_D/a \in \mathbb{R}$), the BPS spectrum also contains a tower of dyons with $n_m = \pm 1$ and arbitrary (integer) n_e. These however correspond to "false flows" breaking down at a regular zero of $Z = n_e a + n_m a_D$ (on the line of marginal stability). The same problem arises for the purely electrically charged massive W-boson.

The paradox can be seen most sharply by starting with a $(0, 1)$ monopole flow and performing a $u \to e^{2i\pi} u$ monodromy around $u = \infty$ (figure 9). Doing this monodromy once should generate a higher dyon with charge $(2, -1)$, doing it twice should yield a $(-4, 1)$ dyon, and so on. However, when circling around $u = \infty$, at a certain point, one arrives at a critical flow passing through the $u = -1$ singularity. When one tries to "pull" the flow through this singularity, a catastrophe occurs: due to the nontrivial monodromy of the magnetic charge around $u = -1$, the flow can no longer end on $u = 1$; instead, past the singularity, it starts to diverge away from the flow just before criticality, and breaks down at the point (on the line of marginal stability) where $Z_{(2,-1)}$ (analytically continued along the flow) becomes zero.

Physically, we do not expect anything really drastic to happen when we vary the moduli at infinity just a little bit, yet we seem to find it can cause a complete breakdown of the solution.

So what is going on here?

[6]The factor $\sqrt{2}$ is due to the conventions used in [18]. We take $\Lambda \equiv 1$.

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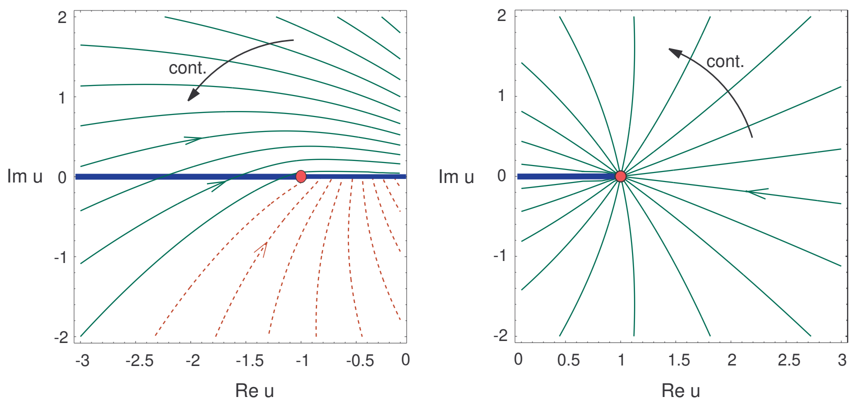

Figure 9: *Right:* Monopole flows in the Seiberg-Witten u-plane for $\mathrm{Re}\,u > 0$. *Left:* continuation of these flows to $\mathrm{Re}\,u < 0$ by rotating the starting point of the flow counterclockwise around $u = \infty$. The fat blue lines are period cuts. The attractor point is the "conifold" point $u = 1$, where the monopole becomes massless. When performing the monodromy around $u = \infty$, a critical flow is encountered (passing through the dyon singularity $u = -1$) beyond which no solution exists: the red dashed lines are "false flows"; they continue beyond the cut (not shown on figure) and crash at a regular zero.

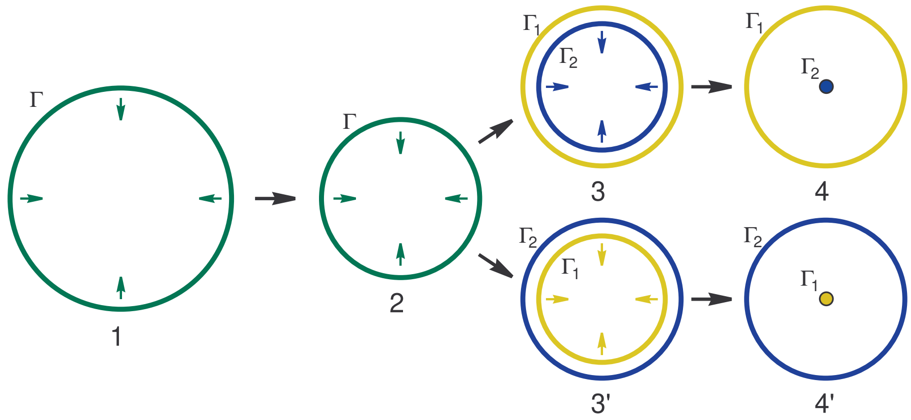

Figure 10: The two possible sequences of events for the inmoving shell of section 5.1, as described in the text; 1: Γ-shell contracts, 2: marginal stability for decay $\Gamma \to \Gamma_1 + \Gamma_2$ is reached at $r = r_{\rm ms}$ and the Γ-particles split, 3: Γ_1-shell stays at $r_{\rm ms}$ and Γ_2 moves further in, 4: final configuration, a Γ_2-center surrounded by a Γ_1-shell, 3',4': same as 3,4 with Γ_1 and Γ_2 interchanged.

6. Resolutions

6.1 Composite configurations

We now turn to the resolution of these puzzles. To get a first hint of what could do the job, consider again the situation described in section 5.1; the suicidal solution produced by an inmoving charged shell of charge Γ. As explained there, we expect that the modulus at spatial infinity and the modulus where Z becomes zero are separated by a line of marginal stability, so we expect the attractor flow to cross this line. Suppose that this is indeed what happens, say at $r = r_{\rm ms}$, for the decay process $\Gamma \to \Gamma_1 + \Gamma_2$, and assume for simplicity that both Γ_1 and Γ_2 have well behaved attractor flows. Then what actually will happen when the shell shrinks is *not* the disaster scenario of section 5.1; instead, when the particle cloud has reached radius $R = r_{\rm ms}$, *the* Γ*-particles will decay into* Γ_1*- and* Γ_2*-particles* (figure 10). Now the Γ_1- and Γ_2-particles cannot *both* continue to move inward, as this would be energetically impossible (the configuration would no longer be BPS because $Z(\Gamma_1)$ and $Z(\Gamma_2)$ acquire different phases on points of the Γ-flow beyond the marginal stability line). Rather, the Γ_1-particles will stay at the marginal stability locus $r = r_{\rm ms}$, while the Γ_2-particles move on, or vice versa. In the first case, when the Γ_2-charges arrive at $r = 0$, we have a BPS configuration (see below) consisting of a Γ_2-charged center surrounded by a Γ_1-shell at $r = r_{\rm ms}$. Outside the shell the fields are given by the Γ-attractor flow, and inside the shell by the Γ_1-attractor flow. In the second case, we have a similar situation, with 1 and 2 interchanged.

To see that such configurations are indeed BPS, let us compute the total energy, say for the first case. The energy in the bulk fields outside the Γ_1-shell is $E_{\rm out} = |Z(\Gamma)|_\infty - (e^U|Z(\Gamma)|)_{r_{\rm ms}}$. The energy of the shell itself is $E_{\rm shell} = (e^U|Z(\Gamma_1)|)_{r_{\rm ms}}$. The energy inside the shell is $E_{\rm in} = (e^U|Z(\Gamma_2)|)_{r_{\rm ms}}$. So the total energy is

$$E_{\rm tot} = |Z(\Gamma)|_\infty + \big(e^U(|Z(\Gamma_1)| + |Z(\Gamma_2)| - |Z(\Gamma_1+\Gamma_2)|)\big)_{r_{\rm ms}} . \tag{6.1}$$

But since precisely at marginal stability, the quantity between brackets is zero, we find indeed $E_{\rm tot} = |Z(\Gamma)|_{r=\infty}$, that is, the configuration is BPS.

Furthermore, when one would move the shell away from $r = r_{\rm ms}$, the quantity between brackets becomes strictly positive, so this configuration is stable under such perturbations.

Another way of seeing this is by considering the force on a test particle of charge $\epsilon\Gamma_1$ at rest in the attractor flow field of a charge Γ_2. This can be derived from (3.2). As shown in appendix A, the result is that this force can be derived from the potential

$$W = 2\,\epsilon\, e^U\, |Z(\Gamma_1)|\, \sin^2\left(\frac{\alpha_1 - \alpha_2}{2}\right) , \tag{6.2}$$

where $\alpha_i = \arg Z(\Gamma_i)$. This potential is everywhere positive, and becomes zero when $\alpha_2 = \alpha_1$, that is, at marginal stability.

It is not difficult to extract the equilibrium radius $r_{\rm ms}$ from the integrated flow equation (3.25). Taking the intersection product of Γ_1 with this equation gives, denoting $Z(\Gamma_i)$ in short as Z_i:

$$2\ \mathrm{Im}(e^{-U}e^{-i\alpha}Z_1) = -\langle\Gamma_1,\Gamma\rangle\,\tau + 2\ \mathrm{Im}(e^{-i\alpha}Z_1)_{\tau=0} . \tag{6.3}$$

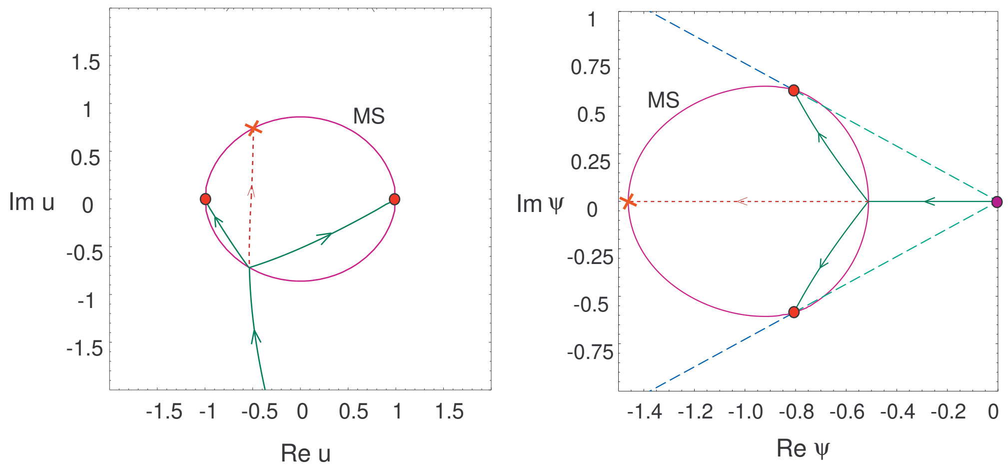

Figure 11: Numerically computed composite flow for a $(2,-1)$-dyon is Seiberg-Witten theory (left) and for the state $|10000\rangle_B$ at the Gepner point of the quintic (right). The purple ellipsoid line is the relevant line of marginal stability. The dotted red line is the false simple attractor flow, crashing on the regular zero of Z indicated by a red cross. The wedge $4\pi/5 < \arg\psi < 6\pi/5$ is indicated by dashed lines.

At $1/\tau = r = r_{\rm ms}$, the left-hand side is zero, so

$$r_{\rm ms} = \frac{\langle\Gamma_1,\Gamma\rangle}{2\ \mathrm{Im}(e^{-i\alpha}Z_1)_{r=\infty}}\,. \tag{6.4}$$

Using $e^{i\alpha} = Z/|Z|$ with $Z = Z_1 + Z_2$ and $\langle\Gamma_1,\Gamma\rangle = \langle\Gamma_1,\Gamma_2\rangle$, this can be written more symmetrically as

$$r_{\rm ms} = \frac{1}{2}\langle\Gamma_1,\Gamma_2\rangle\,\frac{|Z_1+Z_2|}{\mathrm{Im}(\bar{Z}_2 Z_1)}\bigg|_{r=\infty}\,. \tag{6.5}$$

Some interesting consequences of this identity will be discussed further on.

Having arrived at this picture of composite configurations in the approximation of spherical shells, a natural question to ask is whether supergravity also has non-spherical multicenter BPS solutions (with non-parallel charges). We will study this problem in section 7.

6.2 Forked flows

The composite configurations discussed above can be represented by *composite flows* (or "forked flows"): the flow starts as an ordinary Γ-attractor flow, reaches a line of marginal stability, and then splits in a Γ_1-flow and a Γ_2-flow, corresponding to the two possible realization of the state as a charged center surrounded by a charged shell. The total energy of the configuration then equals the sum of the energies associated to each of the constituent flows, that is, for a γ-flow running from i to f, $E = (e^U|Z(\gamma)|)_f - (e^U|Z(\gamma)|)_i$.

Thus the generalization to composite spherically symmetric BPS states simply amounts to the generalization of simple attractor flows to composite attractor flows. Can we find such composite flows for the specific examples discussed in sections 5.1 and 5.2? Fortunately, it turns out we can. As shown in figure 11, the $\Gamma = (2, -1)$ dyon in Seiberg-Witten theory can be realized as a flow splitting in a $\Gamma_1 = (0, 1)$ monopole flow and a $\Gamma_2 = 2(1, -1)$ elementary dyon flow. This corresponds, in the supergravity regime with $\Gamma = N(2, -1)$, N large, to a magnetic core with charge $N(0, 1)$ surrounded by a dyonic shell with charge $2N(1, -1)$, or vice versa. The intersection product of an elementary dyon and a monopole equals 2.

For the quintic example outlined in section 5.1, we find a composite flow ending on two copies of the conifold point (figure 11). In the conventions and notation of [35], the state $|10000\rangle_B$ under consideration has type-IIA D-brane charge $(Q_6, Q_4, Q_2, Q_0) = (2, 0, 5, 0)$, while the charges with vanishing mass at the two conifold point copies under consideration are $(-4, -3, -14, 10)$ and $(6, 3, 19, -10)$,[7] adding up to the required $(2, 0, 5, 0)$. The intersection product of these two charges equals 5.

The appearance of these composite flows is very reminiscent of the appearance of "3-pronged strings" in the "3-1-7 brane picture" of BPS states in $\mathcal{N} = 2$ quantum field theories [20, 21]. This is no coincidence. As explained in section 3.3, in the Seiberg-Witten case for instance, there is an exact map between the attractor flows and the stretched strings of [20]. Similarly, the composite flows arise precisely when the simple geodesic strings fail to exist and the 3-pronged strings take over, and here again there is an exact map between the flows and the strings.

Finally note one could imagine more complex configurations, involving more than one shell, corresponding to more than one flow split. For now, we will stick to the two charge case however.

6.3 Monodromy magic

This picture also offers a nice way to resolve the monodromy puzzle of section 5.2. Consider again N monopole charges at $r = 0$, in a vacuum u_∞ such that the attractor flow is infinitesimally close to the critical one. When we further vary u_∞ counter-clockwise, the flow will pass through the $u = -1$ point, say at $r = r_c$. Placed at this radius, an elementary dyon would be massless, so it could be created there at no cost in energy. And this is precisely what will happen when we continue to rotate u_∞: $2N$

[7] These slightly unnaturally looking values arise because the type-IIA D-brane charges are naturally defined only at large volume (or large complex structure on the IIB mirror). Charges at arbitrary ψ are defined by continuous transport coming from large ψ in the wedge $0 < \arg\psi < 2\pi/5$. This procedure assigns charge $(1, 0, 0, 0)$ to the state with vanishing mass at $\psi = 1$. The states with vanishing mass at the other four copies of the conifold point get charges related to this one by the $\mathbb{Z}_5$ monodromy around the Gepner point [35], which has no reason to have a particularly nice action when expressed in the type-IIA D-brane basis.

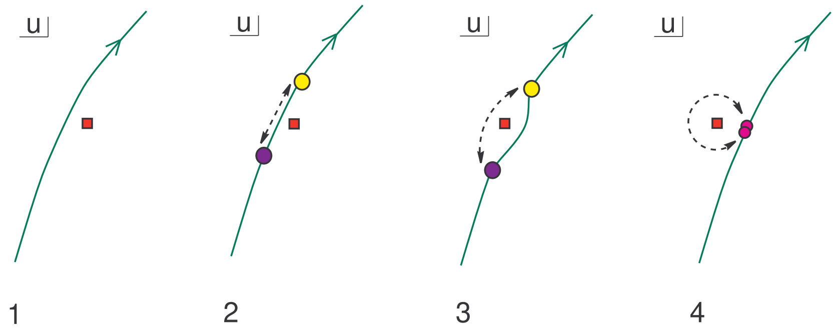

Figure 12: 1: $u(r)$-flow just before criticality (the red rectangle is the point $u = -1$); 2: a virtual monopole-antimonopole pair is created; 3: flow moves beyond $u = -1$; 4: pair annihilates again but leaves two light elementary dyons behind, due to monodromy.

elementary dyons of charge $(1, -1)$ are created! Due to the subtleties associated with monodromy, this is in full agreement with charge conservation. To get some physical feeling for this phenomenon, suppose we manipulate the $u(r)$-field in a certain region of space containing a piece of the surface $r = r_c$, in such way that here the $u(r)$-flow moves from passing just above to passing just below the $u = -1$ singularity in the u-plane (figure 12). Now imagine that just before the move a virtual monopole-antimonopole pair was created, to be destroyed again just after the move, and that the monopole happened to be at $r > r_c$ when the critical trajectory was crossed, while the antimonopole was at $r < r_c$. Then the spacetime trajectory of the monopole-antimonopole pair, mapped to moduli space via $u(r,t)$, encircles the point $u = -1$. Consequently, there is a monodromy on the monopole charge, of which the net result is that we are left with two elementary dyons (with infinitesimal mass) when the monopole-antimonopole pair is destroyed again! If the resulting configuration is energetically favorable (so certainly if it is BPS), it will persist. This gives a physically reasonable mechanism to get the required dyons for the composite BPS state that is supposed to take over when we further rotate u_∞. Once the dyon shell is present, we can continue the monodromy (making the dyon shell massive), till the (composite) flow passes through the $u = 1$ singularity, where the above process is repeated with massless monopoles.[8] In this way we can continue, creating all expected higher dyons.

Note that a local observer, placed in- or outside the sphere $r = r_c$, will not note anything peculiar when the transition takes place. Locally, everything changes perfectly smoothly.

[8] The reader might be puzzled about how our configuration with a dyonic outer shell gets transformed into one with a magnetic outer shell. This can be understood from the discussion of empty holes in section 4.1.2: when approaching the flow passing through $u = 1$, the distance between the dyonic shell and the "enhançon" radius $r = r_*$, where $u = 1$ is reached and below which the monopoles cannot be localized, shrinks to zero. Thus, at the critical flow, the roles of the monopole and dyon shells can be interchanged continuously.

6.4 Marginal stability, Joyce transitions and Π-stability

From (6.5), it follows that when the moduli at infinity approach the line (or, if the dimension of moduli space is larger than one, the hypersurface) of marginal stability for the decay $\Gamma \to \Gamma_1 + \Gamma_2$, the shell radius $r_{\rm ms}$ will diverge, eventually reaching infinity at marginal stability. This gives a nicely continuous 4d spacetime picture for the decay of the state when crossing marginal stability.

Furthermore, (6.5) tells us at which side of the marginal stability hypersurface the composite state can actually exist: since $r_{\rm ms} > 0$, it is the side satisfying

$$\langle \Gamma_1, \Gamma_2 \rangle \, \sin(\alpha_1 - \alpha_2) > 0 \,, \tag{6.6}$$

where $\alpha_i = \arg Z(\Gamma_i)_{r=\infty}$. Sufficiently close to marginal stability, this reduces to

$$\langle \Gamma_1, \Gamma_2 \rangle \, (\alpha_1 - \alpha_2) > 0 \,, \tag{6.7}$$

which is precisely the stability condition for "bound states" of special lagrangian 3-cycles found in a purely Calabi-Yau geometrical context by Joyce! (under more specific conditions, which we will not give here) [22, 32].

Note also that, since the right-hand side of (6.3) can only vanish for one value of τ, the composite configurations we are considering here will actually satisfy

$$|\alpha_1 - \alpha_2| < \pi \,. \tag{6.8}$$

Another immediate consequence of (6.5) is the fact that these composite configurations can only[9] occur for mutually non-local charges, that is, charges Γ_1 and Γ_2 with non-zero intersection product.

If the constituent Γ_i of the composite configuration for which $\langle \Gamma, \Gamma_i \rangle > 0$ can be identified with a "subobject" of the state as defined in [40], the above conditions imply that the phases satisfy the Π-stability criterion introduced in that reference. Though this similarity is interesting, it is far from clear how far it extends. Π-stability is considerably more subtle than what emerges here. On the other hand, we have thus far only considered BPS configurations in a classical, spherical shell approximation, so also on this side the full stability story can be expected to be more complicated. We leave this issue for future work.

7. The general stationary multicenter case

In view of the emergence of composite BPS configuration in the spherical approximation, it is natural to look for more general multicenter solutions. This case is far more involved however. In particular, we have to give up the assumption that the configuration is static, and allow for more general, but still stationary, spacetimes.

[9] at least for asymptotically flat space. For a space asymptotic to $AdS_2 \times S^2$, the situation changes.

7.1 BPS equations

Stationary (single center) BPS solutions of $\mathcal{N} = 2$ supergravity were first studied in [11] from supersymmetry considerations, in a specific space-dependent Ω_0-gauge (essentially the one described at the end of section 3.4). Here we will follow an approach based on the bosonic duality invariant action, similar to the one followed in section 3, and we let Ω_0 depend on position only through the the moduli.

Again, we will use the metric ansatz (3.7), but now with U an arbitrary function of position $\mathbf{x}$, and ω not necessarily zero (but still time independent). We consider only the asymptotically flat case here, that is, $U, \omega \to 0$ when $r \to \infty$.

We will use boldface notation for 3d quantities as explained in section 3.1. The 3d Hodge dual with respect to the *flat* metric δ_{ij} will be denoted by $*_0$, and for convenience we write $\tilde{\omega} \equiv e^{2U}\omega$. It will also turn out to be useful to define the following scalar product of spatial 2-forms $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$:

$$(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}) \equiv \frac{e^{2U}}{1-\tilde{\omega}^2} \int_X \boldsymbol{\mathcal{F}} \wedge \Big[*_0 \widehat{\boldsymbol{\mathcal{G}}} - *_0(\tilde{\omega} \wedge \widehat{\boldsymbol{\mathcal{G}}}\,)\tilde{\omega} + *_0\,(\tilde{\omega} \wedge *_0 \boldsymbol{\mathcal{G}}) \Big] . \tag{7.1}$$

Note that we have $(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}) = (\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{F}})$ and for $\tilde{\omega}$ not too large $(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}) \geq 0$.

With these assumptions and notations, the action (3.1), with the duality invariant electromagnetic action (3.6) substituted in place of the covariant one, becomes, putting $\gamma \equiv \sqrt{4\pi}$ and dropping a total derivative $\sim \Delta U$ from the gravitational action:

$$\begin{aligned} S_{4D} = -\frac{1}{16\pi} \int dt \int_{\mathbb{R}^3} \Big\{ & 2\,\mathbf{d}U \wedge *_0 \mathbf{d}U - \frac{1}{2} e^{4U} \mathbf{d}\omega \wedge *_0 \mathbf{d}\omega + \\ & + 2 g_{a\bar{b}}\, \mathbf{d}z^a \wedge *_0 \mathbf{d}\bar{z}^{\bar{b}} + (\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}) \Big\} . \end{aligned} \tag{7.2}$$

We will derive the BPS equation by "squaring" the action in a way inspired by (3.22). Let α be an arbitrary real function on $\mathbb{R}^3$, denote

$$\mathbf{D} \equiv \mathbf{d} + i \left(\mathbf{Q} + \mathbf{d}\alpha + \frac{1}{2} e^{2U} *_0 \mathbf{d}\omega \right) , \tag{7.3}$$

with $\mathbf{Q}$ as in (2.16), and define the 2-form $\boldsymbol{\mathcal{G}}$ as

$$\boldsymbol{\mathcal{G}} \equiv \boldsymbol{\mathcal{F}} - 2\,\mathrm{Im} *_0 \mathbf{D} \left(e^{-U} e^{-i\alpha} \Omega \right) + 2\,\mathrm{Re}\, \mathbf{D} \left(e^{U} e^{-i\alpha} \Omega\, \omega \right) . \tag{7.4}$$

Then we find for the integrand $\mathcal{L}$ of (7.2), after some calculational effort involving repeated use of the identities (2.12) and (2.13)–(2.15),

$$\begin{aligned} \mathcal{L} = (\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}) - 4 \left(\mathbf{Q} + \mathbf{d}\alpha + \frac{1}{2} e^{2U} *_0 \mathbf{d}\omega \right) \wedge \mathrm{Im} \langle \boldsymbol{\mathcal{G}}, e^U e^{-i\alpha} \Omega \rangle + \\ + \mathbf{d} \Big[2\,\tilde{\omega} \wedge (\mathbf{Q} + \mathbf{d}\alpha) + 4\,\mathrm{Re} \left\langle \boldsymbol{\mathcal{F}}, e^U e^{-i\alpha} \Omega \right\rangle \Big] . \end{aligned} \tag{7.5}$$

Thus if

$$\boldsymbol{\mathcal{G}} = 0\,, \tag{7.6}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \frac{1}{2} e^{2U} *_0 \mathbf{d}\omega = 0\,, \tag{7.7}$$

we have a BPS solution to the equations of motion following from the reduced action (7.2)[10] (we will verify the saturation of the BPS bound below). Now from (7.7) and (7.3), we have $\mathbf{D} = \mathbf{d}$, and (7.6) becomes

$$\boldsymbol{\mathcal{F}} + 2\,\mathbf{d}\ \mathrm{Re}\left(e^{U} e^{-i\alpha}\Omega\,\omega\right) = 2 *_0 \mathbf{d}\ \mathrm{Im}\left(e^{-U} e^{-i\alpha}\Omega\right). \tag{7.8}$$

Since by construction $\mathbf{d}\boldsymbol{\mathcal{F}} = 0$ (away from sources), this implies $\mathbf{d} *_0 \mathbf{d}\ \mathrm{Im}(e^{-U} e^{-i\alpha}\Omega) = 0$, so we can write

$$2\ \mathrm{Im}\left(e^{-U} e^{-i\alpha}\Omega\right) = H\,, \tag{7.9}$$

with H a $H^3(X,\mathbb{Z})$-valued harmonic function on $\mathbb{R}^3$ (possibly with source singularities). If we take the sources to be at positions $\mathbf{x}_i$ with charges Γ_i, where $i = 1, \ldots, N$, then from (7.8) and (3.3), we obtain

$$H = -\sum_{i=1}^{N} \Gamma_i\,\tau_i + 2\ \mathrm{Im}\left(e^{-i\alpha}\Omega\right)_{r=\infty}\,, \tag{7.10}$$

with $\tau_i = 1/|\mathbf{x} - \mathbf{x}_i|$. Defining the 1-form

$$\boldsymbol{\zeta} \equiv -\langle \mathbf{d}H, \Omega\rangle = \sum_{i=1}^{N} Z(\Gamma_i)\,\mathbf{d}\tau_i\,, \tag{7.11}$$

we get from taking intersection products of $\mathbf{d}H$ given by (7.9) with Ω and $D_a\Omega$, and using (2.13)–(2.15):

$$\mathbf{Q} + \mathbf{d}\alpha = e^{U}\,\mathrm{Im}\left(e^{-i\alpha}\boldsymbol{\zeta}\right) = -\frac{1}{2} e^{2U}\langle \mathbf{d}H, H\rangle\,, \tag{7.12}$$

$$\mathbf{d}U = -e^{U}\,\mathrm{Re}\left(e^{-i\alpha}\boldsymbol{\zeta}\right), \tag{7.13}$$

$$\mathbf{d}z^a = -e^{U} g^{a\bar{b}} e^{i\alpha} \bar{D}_{\bar{b}}\boldsymbol{\zeta}\,. \tag{7.14}$$

Using (7.12), equation (7.7) can be rewritten as:

$$*_0\mathbf{d}\omega = \langle \mathbf{d}H, H\rangle\,. \tag{7.15}$$

Equations (7.9) and (7.10) generalize (3.25). Given the sources and the moduli at infinity, they yield the fields $U(\mathbf{x})$, $\alpha(\mathbf{x})$ and $z^a(\mathbf{x})$. Equation (7.15) on the other

[10] We will assume that these solutions also satisfy the equations of motion of the full action without restrictions on the metric, as in the spherically symmetric case, though we did not check this explicitly.

hand gives $\omega(\mathbf{x})$ (up to gauge transformations $\omega \to \omega + \mathbf{d}f$, which can be absorbed by a coordinate transformation $t \to t - f$). Equations (7.13) and (7.14) generalize the flow equations (3.18)–(3.19).

Note that asymptotically for $1/\tau = r \to \infty$, the right-hand side of (7.12) vanishes and $\zeta \to \sum_i Z(\Gamma_i)\mathbf{d}\tau$, implying

$$\alpha \longrightarrow \arg Z(\Gamma) \qquad \text{and} \qquad \zeta \longrightarrow Z(\Gamma)\mathbf{d}\tau \quad \text{when } r \longrightarrow \infty\,, \tag{7.16}$$

where $\Gamma = \sum_i \Gamma_i$. Thus, far from all sources, we have again a simple attractor flow, corresponding to the total charge Γ, as could be expected physically. In particular (7.13) gives $\mathbf{d}U \to -|Z(\Gamma)|d\tau$, with $\tau = 1/r$, establishing the saturation of the BPS bound on the mass:

$$M_{ADM} = |Z(\Gamma)|_{r=\infty}\,. \tag{7.17}$$

In the spherically symmetric case (and in the multicenter case with parallel charges), the above asymptotics become exact, and we retrieve the equations found earlier for those cases. Similarly, close to the center $\mathbf{x}_i$, we have

$$\alpha \longrightarrow \arg Z(\Gamma_i) \qquad \text{and} \qquad \zeta \longrightarrow Z(\Gamma_i)\mathbf{d}\tau_i \quad \text{when } \mathbf{x} \longrightarrow \mathbf{x}_i\,, \tag{7.18}$$

and again we have asymptotically the flow equations for a single charge attractor, as could be expected physically. In particular the moduli at $\mathbf{x}_i$ will be fixed at the Γ_i-attractor point.

The BPS equations of motion for the moduli and the metric obtained here can be seen to reduce to the equations found in [11] in the Ω_0-gauge described at the end of section 3.4, except that we do not find the restricition $\mathbf{dQ} = 0$.

7.2 Some properties of solutions

Consider a multicenter solution, with distinct centers $\mathbf{x}_i$, $i = 1, \ldots, n$, to the BPS equations

$$2\, e^{-U}\, \mathrm{Im}\left(e^{-i\alpha}\Omega\right) = H\,, \tag{7.19}$$

$$*_0\mathbf{d}\omega = \langle \mathbf{d}H, H\rangle\,, \tag{7.20}$$

where

$$H = -\sum_{i=1}^{n} \Gamma_i\, \tau_i + 2\ \mathrm{Im}\left(e^{-i\alpha}\Omega\right)_{r=\infty}\,, \tag{7.21}$$

as derived in the previous section. Acting with $\mathbf{d}*_0$ on equation (7.20) gives

$$0 = \langle \Delta H, H\rangle\,, \tag{7.22}$$

so, using (7.21) and $\Delta\tau_i = -4\pi\delta^3(\mathbf{x} - \mathbf{x}_i)$, we find that for all $i = 1, \ldots, n$:

$$\sum_{j=1}^{n} \frac{\langle \Gamma_i, \Gamma_j\rangle}{|\mathbf{x}_i - \mathbf{x}_j|} = 2\ \mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_{\infty}\,. \tag{7.23}$$

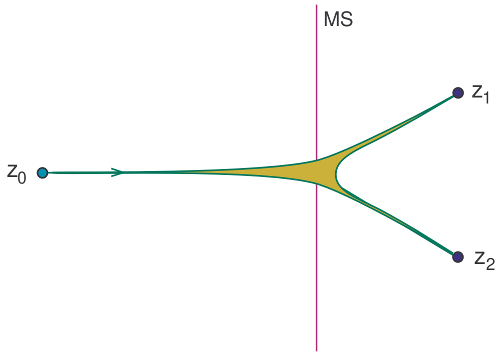

Figure 13: Sketch of the image of $z(\mathbf{x})$ in moduli space for a multicenter solution containing two different charges Γ_1 and Γ_2, with attractor points z_1 resp. z_2, and modulus at spatial infinity z_0. The line labeled "MS" is a (Γ_1, Γ_2)-marginal stability line.

In the particular case of one source with charge Γ_2 at $\mathbf{x} = 0$ and m sources with equal charge Γ_1 at positions $\mathbf{x}_i$, this becomes

$$|\mathbf{x}_i| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2 \, \mathrm{Im}(e^{-i\alpha} Z(\Gamma_1))_\infty}, \tag{7.24}$$

which is equal to the equilibrium distance $r_{\rm ms}$ found in the spherical shell picture, equation (6.4).

In general, the moduli space of solutions to (7.23) will be quite non trivial. Some general properties can be deduced relatively easy however. For instance, in a configuration made of only two different charge types Γ_1 and Γ_2 (distributed over an arbitrary number of centers), the charges of different type, if mutually non local, will be driven to infinite distance from each other when (Γ_1, Γ_2)-marginal stability is approached. This is similar to what we found for the spherical shell case. The stability condition (6.6) reappears as well. If on the other hand the two charges are mutually local (zero intersection), no BPS configuration exists with the two charges separated from each other, unless their phases are equal, that is, at marginal stability (then we can place the charges anywhere).[11]

For configurations made of more charge types, things get more complicated, but we will not go into this here.

Finally, note that because of the asymptotics discussed at the end of the previous section, we can expect the image of the moduli fields into moduli space for a multicenter configuration with only two different charge types to look like a fattened version of the composite flows we introduced earlier to represent the composite

[11] If we consider a spacetime asymptotic to $AdS_2 \times S^2$ instead of the asymptotically flat one we are assuming here, mutually local charges are no longer constrained by (7.23), because there will be an additional factor $\exp[-U(r = \infty)] \equiv 0$ on the r.h.s. of (7.23).

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spherical shell configurations (figure 13). Furthermore, we can expect that the more spherically symmetric the multicenter configuration becomes, the more this fattened version will approach the one-dimensional composite flow. This is similar to what was found for spatial descriptions of dyons in $\mathcal{N}=4$ (effective) quantum field theories [46, 47].

7.3 Angular momentum

It is well known from ordinary Maxwell electrodynamics that multicenter configurations with mutually non-local charges (e.g. the monopole-electron system) can have intrinsic angular momentum even when the particles are at rest. The same turns out to be true here.

We define the angular momentum vector $\mathbf{J}$ from the asymptotic form of the metric (more precisely of ω) as [15]

$$\omega_i = 2\,\epsilon_{ijk}\,J^j\,\frac{x^k}{r^3} + O\left(\frac{1}{r^3}\right) \quad \text{for } r \longrightarrow \infty\,. \tag{7.25}$$

Plugging this expression in (7.20) and using (7.21) and (7.23), we find

$$\mathbf{J} = \frac{1}{2}\sum_{i<j}\langle\Gamma_i,\Gamma_j\rangle\,\mathbf{e}_{ij}\,, \tag{7.26}$$

where $\mathbf{e}_{ij}$ is the unit vector pointing from $\mathbf{x}_j$ to $\mathbf{x}_i$:

$$\mathbf{e}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|}\,. \tag{7.27}$$

Just like in ordinary electrodynamics, this is a “topological” quantity: it is independent of the details of the solution and quantized in half-integer units (more precisely, when all charges are on the z-axis, $2J_z \in \mathbb{Z}$).

The appearance of intrinsic configurational angular momentum implies that quantization of these composites will have some non-trivial features.

8. Conclusions

We have shown the emergence of some puzzles and paradoxes arising when one tries to construct four-dimensional low energy effective supergravity solutions corresponding to certain BPS states in type-II string theory compactified on a Calabi-Yau manifold, and demonstrated how these can be resolved by considering composite and extended configurations. We made connections to the enhançon mechanism, the 3-pronged string picture of QFT BPS states, Π-stability and Joyce transitions of special lagrangian manifolds. The problem was analyzed in a spherical shell approximation and by considering multicenter BPS solutions.

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There are quite some problems however, new and old ones, we did not touch upon. The most prominent one is that we did not analyze to what extent these states really exist as BPS bound states in the full quantum theory. It seems quite likely that we now face the opposite problem we started with: instead of too little, we might now have too many possible solutions. In view of the nontriviality of quantum mechanics with mutually non-local charges, it is not unconceivable that a proper semiclassical treatment would eliminate some of these spurious solutions. But even at the classical level the existence issue is not completely settled. We did not show for instance that all solutions to (7.23) actually lead to well-behaved BPS solutions to the equations of motion; the same phenomenon causing the breakdown of some naively expected spherically symmetric solutions, namely hitting a zero of the central charge, could cause naively expected solutions to break down in this more complicated setting as well.

In this setup it seems also quite possible that a certain charge can have several different realizations as a BPS solution in a given vacuum, for example both as a single center and as a two center configuration. Crossing a line of marginal stability could then cause one realization to disappear, while leaving the other intact. The D-brane analog of this would presumably be a "jump" in its moduli space. This brings us to another interesting open question: is there a connection between D-brane moduli spaces and supergravity solution moduli spaces? And could those solution moduli spaces (for asymptotically flat or $AdS_2 \times S^2$ spacetimes) teach us something about black hole entropy?

It could also be worthwhile to further explore the relation with Π-stability, briefly mentioned in section 6.4.

Finally, this and other recent work [27, 48] illustrates an apparently recurrent theme in string theory: the resolution of singularities by creation of finitely extended D-brane configurations. It would be interesting to find out what the dielectric, non-commutative D-brane effects of [48] can teach us about the states described in this paper.

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A. Potential for a test charge

From (3.2), it follows that the lagrangian (with respect to the time coordinate t) for a test charge Γ_t at rest in the attractor flow field of a charge Γ is (denoting $Z(\Gamma_t)$ in short as Z_t, and similarly for the other quantities involved)

$$L = -e^U |Z_t| - \frac{\sqrt{\pi}}{\gamma} \langle \Gamma_t, \mathcal{A}_0 \rangle \,, \tag{A.1}$$

where $\mathcal{A}_0$ is obtained from (3.9):

$$\partial_i \mathcal{A}_0 = \frac{\gamma}{\sqrt{4\pi}} \, e^{2U} \, \partial_i \tau \, \widehat{\Gamma} \,. \tag{A.2}$$

From (3.24), we get

$$\Gamma = i \, \partial_\tau \left(e^{-U} \, e^{-i\alpha} \Omega \right) + \text{c.c.} \,, \tag{A.3}$$

so, using (2.15) and (as shown in section 3.4) $Q_\tau + \dot{\alpha} = 0$:

$$\Gamma = -i \, e^{-U} \, \dot{U} \, e^{-i\alpha} \Omega + i \, e^{-U} e^{-i\alpha} D_a \Omega \, \dot{z}^a + \text{c.c.} \,, \tag{A.4}$$

hence from (2.12) and again (2.15):

$$\widehat{\Gamma} = -e^{-U} \, \dot{U} \, e^{-i\alpha} \Omega - e^{-U} e^{-i\alpha} D_a \Omega \, \dot{z}^a + \text{c.c.} \tag{A.5}$$

$$= -e^{-2U} \partial_\tau \left(e^U \, e^{-i\alpha} \Omega \right) + \text{c.c.} \,. \tag{A.6}$$

Therefore

$$\partial_i \left(\frac{\sqrt{\pi}}{\gamma} \langle \Gamma_t, \mathcal{A}_0 \rangle \right) = \frac{1}{2} e^{2U} \, \partial_i \tau \langle \Gamma_t, \widehat{\Gamma} \, \rangle \tag{A.7}$$

$$= -\partial_i \left(e^U \, \mathrm{Re} \left(e^{-i\alpha} Z_t \right) \right) , \tag{A.8}$$

and thus (up to a constant)

$$L = -e^U |Z_t| + e^U \, \mathrm{Re} \left(e^{-i\alpha} Z_t \right) \tag{A.9}$$

$$= -e^U |Z_t| \left(1 - \cos(\alpha_t - \alpha) \right) \tag{A.10}$$

$$= -2 e^U |Z_t| \sin^2 \left(\frac{\alpha_t - \alpha}{2} \right) .$$

The force on the test particle is $F_i = \partial_i L$, so we find for the force potential, as announced in section 6.1:

$$W = 2 e^U |Z_t| \sin^2 \left(\frac{\alpha_t - \alpha}{2} \right) . \tag{A.11}$$

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